Continuous Centrifugation in a Disk Centrifuge

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The design of conical disk centrifuges and interpretation of data therefrom is far from being straightforward. Smith (*3*) and Ambler (*1*) discuss semiempirical methods used in certain applications by centrifuge manufacturers in estimating capacity of a machine. Maloney (*2*) in his review of twenty-three references states that "of the recognized Unit Operations, this one has been the subject of few articles." He finally concludes that the engineering schools in the country are in excellent position to help in developing a sound theoretical background for this unit operation.

Theoretical solutions to the problem of conical disk centrifuges have been found and tested, and it is the purpose of this paper so to report.

KINETIC MECHANISM OF CENTRIFUGATION

In Figure 1 there is shown a schematic diagram of the stack of cones contained in a conical-disk type of centrifuge. The N cones of inner radius R_1 and outer radius R_2 form an angle θ with the vertical and enclose $N - 1$ spaces of thickness s. Feed slurry is fed as indicated, and during the course of its progress up through the $N - 1$ spaces it is centrifuged with angular velocity ω radian/sec. into two cuts, the dilute top cut and the concentrated bottom cut.

In the kinetic mechanism of centrifugation, it is assumed for the moment that the feed slurry contains fluid of density ρ_f and viscosity η and particles of diameter D and density ρ_s, where

$$\rho_s > \rho_f$$

The feed slurry approaches its angular velocity, ω, as it enters the $N - 1$ spaces and moves at the flow rate q or lineal velocity v up through each space.

A given particle, say P, moves laterally with a velocity v_s relative to the fluid. The components of this velocity are v_s' and v_s'', wherein the contribution of gravity is neglected because of its negligible effect in strong centrifugal fields. It is immediately apparent that whether the particle P moves up or down through a space is determined by whether $v - v_s'$ is positive or negative. Negative values of this quantity are associated with low capacities and are of no particular interest here, although subsequent mathematical treatment includes this case as well as that for positive values.

On the assumption that the particle P moves up through a space, then its trajectory involves an increase along the x coordinate as its position along the r coordinate decreases in value.

Any particle P entering at B will appear in the bottom cut if its trajectory does not cross a vertical line drawn through point A; otherwise the particle will appear in the top cut.

If the trajectories are relatively flat, as shown in Figure 1, there will be a limiting trajectory passing through point A. If the trajectories have much more curvature than that shown in Figure 1, then there will be a limiting trajectory which is tangent to a vertical line drawn through A. In any event the limiting trajectory also passes through the entrance B at some point x, say x_0. The fraction of solids from the feed recovered in the bottom cut is $(s - x_0)/s$ and this fraction can be calculated if the mathematical expression for the trajectory under consideration is known.

RESULTS

The mathematical details are contained in the Appendix. Suffice it to say here that two solutions have been developed for the case in which Stokes free settling in a centrifugal field is applicable. The feed slurry and cuts are assumed sufficiently dilute that hindered settling does not play an important role.

The first solution is

$$\frac{s - x_0}{s} = \frac{\cos \Theta}{s}\left\{\frac{b}{2}\ln\left[\left(\frac{b + R_2}{b - R_2}\right)\cdot\left(\frac{b - R_1}{b + R_1}\right)\right] - (R_2 - R_1)\right\} \quad (1)$$

which is an approximate analytical solution in the sense that in order to avoid insuperable mathematical difficulty the approximate expression for v, i.e.,

$$v \simeq \frac{q}{2\pi rs} \quad (2)$$

has been used. The dimensionless groups $\cos\theta$, b/s, R_2/s, R_1/s, b/R_2 and b/R_1 determine the recovery efficiency of the centrifugation.

Equation (1) is plotted in Figure 2 for the case of a Merco centrifuge with specifications as follows: $N = 24$, $\theta = 45°$, $s = 0.16838$ cm., $R_1 = 5.12$ cm., $R_2 = 7.62$ cm., $\omega = 859.85$ radian/sec. (8,210 rev./min.). Of course, θ, s, R_1, and R_2 are the only variables that need be known for plotting. It is obvious from this figure that until b reaches a value of 21.7 (approximately) all particles are recovered. When b is increased beyond this value, losses set in accordingly until the recovery finally approaches zero as b approaches infinity.

The second solution is a numerical solution to the differential equation which has been converted to finite difference form thus

$$\frac{x_{n+1} - x_n}{r_{n+1} - r_n} = \frac{\cos \Theta}{1 - \frac{6b^2}{r_n^2}\left[\frac{x_n}{s} - \frac{x_n^2}{s^2}\right]} \quad (3)$$

No simplifying assumptions have been made concerning v. This solution is also plotted in Figure 2.

It is obvious from Figure 2 that the analytical solution is a fairly good approximation to the numerical solution, particularly at the smaller values of b.

The solutions may be applied to a feed having a distribution of particle sizes. One simply plots the fraction by weight vs. particle-size distribution curve. For a selected value of D on this curve, there is a corresponding point on the curve of Figure 2. The value of $(s - x_0)/s$ associated with the latter point multiplied by the selected weight fraction from the first curve gives the fractional recovery of size D. By repeating this procedure, one may tabulate enough values of $[(\text{wt. fraction})\cdot(s - x_0)/s]$ vs. D so as to form a plot of these and then determine the area under the curve which is the total fractional recovery for the mixture of particle sizes. This procedure has been tested on a pilot run with a Merco centrifuge of the foregoing specifications. In this run $\Delta\rho = 1.988 - 0.945 = 1.043$ g./cc., $\eta = 0.03$ poise, $Q = 2.5$ gal./min. total feed rate to centrifuge. It can be shown that

$$q = \frac{\frac{o}{V}}{\frac{o}{V} + 1} \cdot Q \cdot 2.743 \text{ cc./sec. passage}$$

Also

$$d = D \cdot 10^4$$

By material balance one can also show that

$$\frac{o}{V} = \frac{C_B - C_F}{C_F - C_T}$$

In the run

$$C_F = 46.17 g/L$$

$$C_T = 0.274 g/L$$

$$C_B = 202.3 g/L$$

The feed-particle-size analysis and calculated values of b along with $(s - x_0)/s$ from Figure 2 and the calculated value of

$$\left[\frac{s - x_0}{s}\right][\text{wt. fraction}]$$

are tabulated in Table 1, which shows that the predicted recovery of particles in the underflow from the feed is 99.997%. The actual measured recovery in the run was 99.935%, which suggests pretty fair agreement between theory and practice.

One might wonder how particles migrate back down the upper cone surface to appear in the bottom cut once they arrive at the cone surface. The answer lies in the fact that, at the two extremities of s, the value of v is zero, while toward the center of s, the value of v reaches a maximum. Thus a particle of diameter D migrates up through a space until $v - v_s'$ goes through zero and becomes negative, when it begins to migrate downward to appear as the bottom cut. Again the mathematical details appear in the Appendix, but the fraction of s occupied by y, the thickness of the sludge stream flowing down the top cone wall, is

$$\frac{y}{s} = \frac{1 - \left(1 - \frac{2}{3}\left[\frac{r \sin \Theta}{b}\right]^2\right)^{1/2}}{2} \quad (4)$$

The film thickness is determined by the dimensionless groups, $\sin \theta$ and r/b. While this treatment is valid only for small values of y, it is valuable in that it shows, for example, that in order to keep y small for a given centrifuge and

Table 1

Analysis of Feed and Predicted Recovery

d, μ	Fraction by wt. of particles in feed	Calculated values of b	$\left[\frac{s - x_0}{s}\right]$ Fig. 2 by numerical integration	$\left[\frac{s - x_0}{s}\right]\left[\begin{matrix}\text{Fraction}\\ \text{by wt.}\end{matrix}\right]$
1	0.0000530	27.9	0.560	0.0000297
2	0.00141	14.0	1.00	0.00141
3	0.00660	9.31	1.00	0.00660
4	0.0173	6.98	1.00	0.0173
5	0.0366	5.59	1.00	0.0366
6	0.0669	4.65	1.00	0.0669
7	0.111	3.99	1.00	0.111
8	0.161	3.49	1.00	0.161
9	0.143	3.10	1.00	0.143
10	0.131	2.79	1.00	0.131
11	0.0891	2.54	1.00	0.0891
12	0.0876	2.33	1.00	0.0876
13	0.0745		1.00	0.0745
14	0.0188		1.00	0.0188
15	0.0117		1.00	0.0117
16	0.00710		1.00	0.00710
>16	0.0363		1.00	0.0363

The predicted recovery = 1.0000000 − 0.0000297 = 0.9999703.

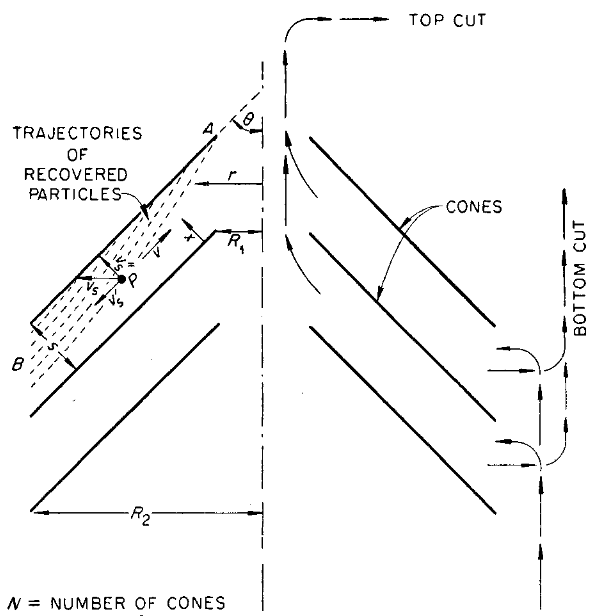

Fig. 1. Schematic drawing of cones.

particle size, one must operate with large b. Also, as one passes from R_2 to R_1, the film thickness decreases for a given particle size.

Ambler (*1*) empirically modified the theory for a simple rotating cylinder centrifuge with end caps to try to make it conform to the operation of a conical disk centrifuge. His result is

$$Q = \frac{2\Delta\rho D^2(N - 1)\pi\omega^2(R_2^3 - R_1^3)}{27\eta \tan \Theta} \quad (5)$$

This formula according to the author is limited to calculating a value of D for the smallest particle that will be recovered. Accuracy is not claimed for the formula and experience in its application is emphasized. A direct comparison with the formula developed in this paper is not obvious.

NOTATION

b = $\left(\frac{9q\eta \sin \Theta}{s\pi\Delta\rho D^2\omega^2}\right)^{1/2}$, cm.
C = concentration, g. solid/liter slurry
D = particle diameter, cm.
d = particle diameter, μ
F = force, g.-force
g_c = 980.665 (g.-mass)(cm./g.-force)(sec.^{2})
N = total number of cones in centrifuge
o = overflow rate, gal./min.
p = pressure, g./sq. cm.
q = flow rate through a single passage, cc./sec. of passage
Q = total slurry feed rate to centrifuge, gal./min.
r = radial position measured normal to axis of centrifuge, cm.
R = radius of top or bottom of a cone, cm.
s = distance between cones, cm.
t = time, sec.
v = fluid velocity at a point in a passage, cm./sec.
v_s = particle velocity relative to fluid, cm./sec.
v_s', v_s'' = components of v_s, cm./sec.
V = underflow rate, gal./min.
x = lineal distance normal to and measured from top side of a cone, cm.
y = sludge film thickness, cm.

Greek Letters

ρ = density, g./cc.
θ = cone angle
η = viscosity, poise (g./cm.)(sec.)
ω = angular velocity, radians/sec.
ϕ = lineal distance measured along cone, cm.

Subscripts

1 = top end of cone, or the first in a series
2 = bottom end of cone, or the second in a series
s = solid phase
f = fluid phase
n = independent variable in finite differences
o on x = x at B in Figure 2 for the limiting trajectory
F = feed
T = top cut or overflow
B = bottom cut or underflow
R = used with a force due to fluid resistance in Stokes settling
S = used with a force due to a centrifugal field

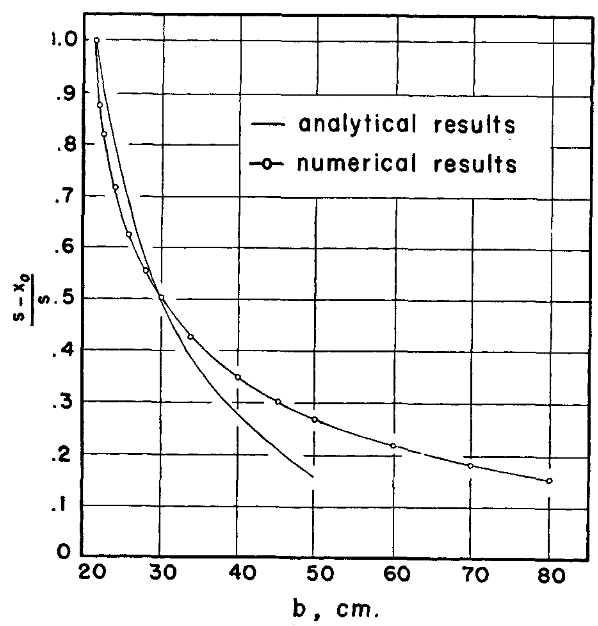

Fig. 2. Solutions to the equations governing centrifugation.

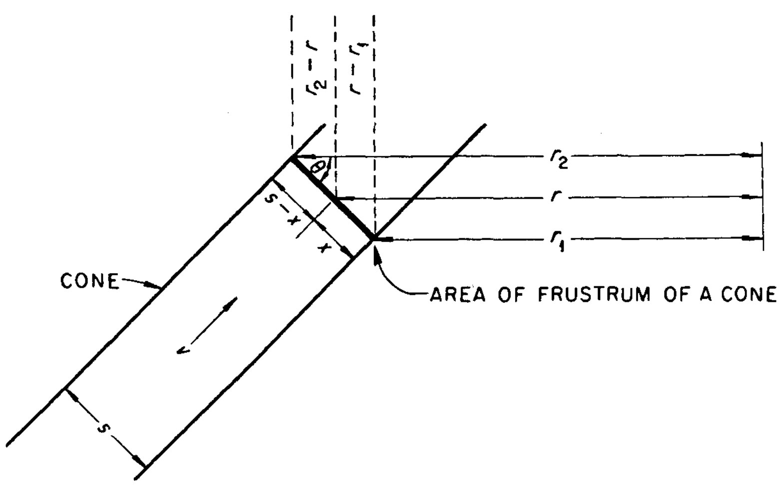

Fig. 3. The cross area of the annular space.

LITERATURE CITED

1. Ambler, C. M., *Chem. Eng. Progr.*, **48**, 150 (1952).
2. Maloney, J. O., *Ind. Eng. Chem.*, **38**, 24, 37 (1946).
3. Smith, J. C., *ibid.*, **39**, 474 (1947).

APPENDIX

Fractional Recovery of Particle Size D—Approximate Analytical Solution

The particle velocity relative to cone wall along x is defined as

$$\frac{dx}{dt} = v_s'' \quad (A1)$$

and along r is defined as

$$\frac{dr}{dt} = v_s - v \sin \Theta \quad (A2)$$

The ratio of Equation (A1) to (A2) is

$$\frac{dx}{dr} = \frac{v_s''}{v_s - v \sin \Theta} \quad (A3)$$

The problem is to express v_s'', v_s, and v as functions of x and r so that integration of Equation (A3) becomes possible. To simplify integration, v may be approximated by the ratio of q to the cross area through which the slurry flows. The cross area under consideration is indicated in Figure 3. The area is

$$\pi(r_2 + r_1)s \tag{A4}$$

but

$$\frac{r_2 - r}{s - x} = \cos \Theta; \qquad \frac{-r + r_1}{x} = -\cos \Theta \tag{A5}$$

adding Equation (A5) shows that

$$r_2 + r_1 = 2r + (s - 2x) \cos \Theta \tag{A6}$$

Therefore

$$v = \frac{q}{\pi[2r + (s - 2x) \cos \Theta]s} \tag{A7}$$

and since

$$2r \gg (s - 2x) \cos \Theta$$

one finds that

$$v \simeq \frac{q}{2\pi rs} \tag{A8}$$

The effective force operating on a spherical particle of diameter D in a centrifugal field is

$$g_c F_S = \frac{\pi}{6} \Delta\rho D^3 \omega^2 r \tag{A9}$$

According to Stokes's Law for free settling, the force resisting motion of the particle through the fluid is

$$g_c F_R = 3\pi\eta D v_s \tag{A10}$$

At equilibrium the two forces F_S and F_R are equivalent if one assumes negligible the force due to the rate of change of momentum of the particle. Thus

$$v_s = \frac{\Delta\rho D^2 \omega^2 r}{18\eta} \tag{A11}$$

Also,

$$v_s'' = \frac{\Delta\rho D^2 \omega^2 r}{18\eta} \cos \Theta \tag{A12}$$

The differential equation (3A) becomes

$$\int_{x_0}^{s} dx = -\cos \Theta \int_{R_2}^{R_1} \frac{r^2\, dr}{b^2 - r^2} \tag{A13}$$

where

$$b = \left(\frac{9q\eta \sin \Theta}{s\pi \Delta\rho D^2 \omega^2}\right)^{1/2} \tag{A14}$$

and the solution is

$$\frac{s - x_0}{s} = \frac{\cos \Theta}{s} \left\{ \frac{b}{2} \ln \left[\left(\frac{b + R_2}{b - R_2}\right) \cdot \left(\frac{b - R_1}{b + R_1}\right) \right] - (R_2 - R_1) \right\} \tag{A15}$$

Sludge Film Thickness for Particle Size D

The definition of laminar viscosity is

$$g_c \frac{\text{force}}{\text{area}} = \eta \frac{dv}{dx} \tag{A16}$$

If the lineal distance along the surface of a cone normal to x is designated as ϕ, then over an elemental section $d\phi$ there will be a corresponding pressure drop dp. If s is small:

$$g_c \text{ force} \simeq \frac{dp}{2} (s - 2x) 2\pi r \tag{A17}$$

and

$$\text{area} = 2\pi r\, d\phi \tag{A18}$$

Substituting in Equation (16), one has

$$g_c \frac{1}{2} \frac{dp}{d\phi} \int_0^x (s - 2x)\, dx = \eta \int_0^v dv \tag{A19}$$

or

$$g_c \frac{1}{2} \frac{dp}{d\phi} [sx - x^2] = \eta v \tag{A20}$$

For small s one has that v rises to its maximum value at

$$x = \frac{s}{2} \tag{A21}$$

From Equation (A20) then

$$g_c \frac{1}{2} \frac{dp}{d\phi} \frac{s^2}{4} = \eta v_{max} \tag{A22}$$

The ratio of Equation (A20) to (A22) produces the result:

$$v = 4 v_{max} \frac{sx - x^2}{s} \tag{A23}$$

For small s

$$q = 2 \cdot 2\pi r \int_0^{s/2} v\, dx \tag{A24}$$

Substituting A23 in A24 and integrating gives

$$q = v_{max} \frac{4\pi r}{3} s \tag{A25}$$

Eliminating v_{max} between Equations (A23) and (A25), one finds the velocity distribution in the annulus to be

$$v = \frac{3q}{\pi r} \frac{sx - x^2}{s^3} \tag{A26}$$

The free settling velocity v_s is

$$v_s = \frac{\Delta\rho D^2 \omega^2 r \sin \Theta}{18\eta} \tag{A27}$$

At some distance,

$$x = y \tag{A28}$$

the velocities v and v_s become equal and Equations (A27) and A25) may be combined to show that

$$\frac{y}{s} = \frac{1 - \left[1 - \frac{2}{3}\left(\frac{r}{b} \sin \Theta\right)^2\right]^{1/2}}{2} \tag{A29}$$

The thickness y is that of the sludge film flowing down the cone wall. Equation (A29) may also be derived by setting

$$\frac{dr}{dx} = \cos \Theta; \qquad r = R_1$$

in the differential equation.

FRACTIONAL RECOVERY OF PARTICLE SIZE D, NUMERICAL SOLUTION

If one employs Equations (A11), (A12), and (A26) to eliminate the velocities from Equation (A3), he finds that

$$\frac{dx}{dr} = \frac{\cos \Theta}{1 - \frac{6b^2}{r^2}\left[\frac{x}{s} - \frac{x^2}{s^2}\right]} \tag{A30}$$

This equation is difficult to solve in closed analytical form. It may however be solved by numerical methods if written in finite difference form thus

$$\frac{x_{n+1} - x_n}{r_{n+1} - r_n} = \frac{\cos \Theta}{1 - \frac{6b^2}{r_n^2}\left[\frac{x_n}{s} - \frac{x_n^2}{s^2}\right]} \tag{A31}$$

It will be noted from Equation (A30) that if

$$\frac{dr}{dx} = 0; \qquad r = R_1$$

then

$$1 - \frac{6b^2}{R_1^2}\left[\frac{x}{s} - \frac{x^2}{s^2}\right] = 0 \tag{A32}$$

or in the finite difference case

$$\frac{x_n}{s} = \frac{1 + \left[1 - \frac{2}{3}\left(\frac{R_1}{b}\right)^2\right]^{1/2}}{2} \tag{A33}$$

Equation (A33) was the starting point for numerical calculation of a trajectory. By substituting s, R_1, and b, one can calculate the point x_1 on the vertical line drawn through A of Figure 2. This is the point of tangency of the limiting trajectory. If one substitutes x_1, R_1, b, s and Θ in Equation (A31) and selects an interval for Δx, then r_2 can be calculated. This process is simply repeated to obtain r_3, r_4, etc.

For

$$\Delta x = -0.09935; \qquad b = 80$$

the fractional recovery was 0.1802. When Δx was cut in half, the recovery became 0.1728. When halved three more times, the recovery became 0.1660, 0.1615 and 0.1610 respectively. The latter calculation for this and the other values of b lead to the data which form the basis of the plot in Figure 2.